

A NOTE ON QUANTIZATION IN THE PRESENCE OF GRAVITATIONAL SHOCK WAVES

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ABSTRACT. We study the quantization of a free scalar field when the background metric satisfies Einstein's equations and develops gravitational shock waves.

1. INTRODUCTION.

The general notion of “singularity”, although not always precisely defined and having different meanings in different situations, plays a crucial role in many physical theories. Broadly speaking, one expects that singularities will form in regimes where the system undergoes extreme dynamic conditions, e.g., turbulence in Fluid Dynamics or gravitational collapse in General Relativity (GR).

One particular notion of singular behavior frequently encountered is that of a shock wave. Roughly, it corresponds to a discontinuity in the solutions of the equations of motion — in particular, it implies that such solutions exist only in a generalized sense. In the case of Einstein's theory of gravity, shock wave discontinuities happen in the first derivatives of the metric (see definition 2.8). It follows that the spacetime does not carry a “smooth geometry,” with the curvature tensor (which depends on second derivatives of the metric) being meaningfully defined only in a distributional sense.

Although in a different context, physicists have been dealing with singular spacetimes for quite a long time, with Hawking's singularity theorems [HE] and its ramifications (Cosmic Censorship, Penrose Inequality, etc.) being at the core of such developments. It is widely believed that a successful quantum theory of gravity will resolve many, if not all, of the difficulties and puzzles that arise in singular backgrounds. Unfortunately, despite much of the progress that has been witnessed in the last few decades, such a theory is not yet available. This does not mean, however, that we cannot learn something about quantum effects in curved spacetimes, singular ones included. In the range where curvature effects cannot be neglected but are still far from the Planck scale, the powerful (if yet difficult) machinery of Quantum Field Theory (QFT) in curved backgrounds is available to us. To this day, some of the best hints of what the long sought quantum theory of gravity might look like come from the study of quantum fields over a non-flat background that satisfies Einstein's equations. In fact, the ability of reproducing the black hole temperature — discovered by Hawking via a careful application of QFT on a background that undergoes classical gravitational collapse [H] — is often regarded as the first test for any theory attempting to quantize the gravitational field. This, of course, is not different than many other instances in Physics where semi-classical formulations are useful in providing insight into what the full quantum theory is, leading in this way to fruitful directions of inquiry.

Furthermore, one should not have to wait until a full-fledged theory of quantum gravity is in place in order to understand interesting physical phenomena, which involve quantum effects in a background where the singularities are still amenable to a fully classical treatment. This paper is a step in this direction. We shall study the quantization of a free scalar field over a spacetime where gravitational shock waves are formed. Einstein's equations will have to be defined in a suitable weak sense, and some standard arguments, like the construction of propagators, adapted to this weaker setting. A similar aspiration, namely, devise a quantization scheme that can potentially incorporate singular background data, was explored in [AD].

The tools we shall employ are, in a sense, not new. Most of the constructions go back to the work of Lichnerowicz on tensor distributions [L1] (see [D] for similar constructions and generalizations). These have the convenience of being simultaneously suited to the study of shock waves, on one hand [L2], and to the quantization of fields in curved — although smooth — spacetimes, on the other hand [L3]. Not surprisingly, many of the arguments here presented consist of carefully checking that the results of [L3] carry onto the framework of shock waves. As such arguments can be done in charts, our point of view will be purely local. Generalizations to a global setting are possible, provided that further conditions are taken into account; we briefly comment on this at the end.

Although this paper focuses uniquely on mathematical aspects, we stress that the study of gravitational shock waves in general, and corresponding scenarios where quantum effects might become important in particular, has attracted significant attention in the Physics community (see e.g. [DP1, DP2, DP3, HS, 'tH, HU, VV] and references therein), hence the importance of laying out its mathematical foundations.

Notation 1.1. (i) $\text{vol}(g)$ denotes the volume form of the metric g , and $|g|$ its determinant (in a local coordinate patch); (ii) if $\Omega \subseteq M$, M a C^ℓ manifold, $\mathcal{D}_{k,p}(\Omega)$ denotes the space of p -tensors of class C^k , $0 \leq k \leq \ell$, with compact support in Ω , i.e., test tensors in Ω ; we shall use the given metric to identify covariant and contra-variant tensors, hence referring simply to “ p -tensors”; (iii) $\mathcal{D}'_{k,p}(\Omega)$ is the space of continuous real-valued forms on $\mathcal{D}_{k,p}(\Omega)$, where continuity is understood in the sense of the theory of distributions [S]; (iv) we sometimes write $\mathcal{D}_p(\Omega)$ when the differentiability is clear from the context, although for the most part it will suffice to deal with elements in $\mathcal{D}_{0,p}(\Omega)$ and its dual; (v) $\langle v, u \rangle$ means $u \in \mathcal{D}_{k,p}(\Omega)$ evaluated at $v \in \mathcal{D}'_{k,p}(\Omega)$, whereas $\langle u_1, u_2 \rangle_g$ denotes the inner product between the p -tensors u_1 and u_2 , although, when the distributions arise from locally integrable tensors, we naturally identify $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_g$; (vi) we shall denote by T' the tensor distribution defined by a locally integrable tensor T ; (vii) ∇ will denote covariant differentiation with respect to the metric g . (viii) $[\alpha|\gamma|\beta]$ indicates anti-symmetrization in the indices α and β .

Hypotheses in the form “Assumption X.X” (e.g., Assumption 2.1) are assumed throughout, therefore we do not state them in the theorems and definitions.

2. GRAVITATIONAL SHOCK WAVES.

We start recalling some definitions and fixing out notation. Let M be an oriented 4-dimensional differentiable manifold of class C^2 and piecewise C^4 , and $\Omega \subset M$ a contractible open subset. Whenever coordinates are employed, it is implicitly assumed that Ω is taken small enough as to belong to the domain of a coordinate patch. Let Σ be a regular hypersurface in Ω which is given locally as the level set $\{\varphi = 0\}$ of a C^1 function $\varphi : \Omega \rightarrow \mathbb{R}$; saying that Σ

is regular then means $d\varphi \neq 0$ on Σ . We suppose that Σ partitions Ω into two disjoint domains Ω^+ and Ω^- given by $\{\varphi > 0\}$ and $\{\varphi < 0\}$, respectively.

Definition 2.1. Let T be a p -tensor on Ω , and assume that T is of class C^0 over Ω^+ and Ω^- . T is *regularly discontinuous across*¹ Σ if, when $\varphi \rightarrow 0^+$ (i.e., φ tends to zero through positive values) (resp. $\varphi \rightarrow 0^-$), T converges uniformly to a p -tensor T^+ (resp. T^-) defined on Σ . The *discontinuity*² $\{\{T\}\}$ of T is the p -tensor on Σ defined by $\{\{T\}\} = T^+ - T^-$.

Let g be Lorentzian metric³ of class C^0 and piecewise C^2 on M . We fix Ω , φ and Σ once and for all, and henceforth assume for the rest of the paper:

Assumption 2.1. (i) φ is C^3 on Ω^+ and Ω^- , with second and third derivatives regularly discontinuous across Σ ; (ii) g is C^0 on Ω , C^2 on Ω^+ and Ω^- , and has first and second derivatives regularly discontinuous across Σ .

Unless stated otherwise, it will be assumed in this section that coordinates $\{x^\alpha\}_{\alpha=0}^3$ are adapted to Σ , what means that $x^0 = \varphi$ and ∂_i is tangent to Σ , $i = 1, 2, 3$. From now on, Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

Definition 2.2. In coordinates $\{x^\alpha\}_{\alpha=0}^3$ adapted to Σ , the metric components g_{ij} will be called *physical components with respect to Σ* , while $g_{0\alpha}$ will be called *gauge* (or *non-physical components with respect to Σ*). A change of coordinates $x'^\alpha = x'^\alpha(x^0, \dots, x^3)$ is called a *change of gravitational gauge with respect to Σ* , or *change of gauge* for short, if it is the identity on Σ , and the the values of $g_{\alpha\beta}|_\Sigma$ as well as of $\partial_0 g_{ij}|_\Sigma$ are invariant under this change.

Remark 2.3. Despite the familiar terminology, the reader should not be led to think that Σ is a space-like hypersurface. In fact, for the case of interest in this paper, Σ will be light-like.

The necessity of treating g_{ij} and $g_{0\alpha}$ differently comes from the well-known fact that GR has a gauge freedom due to the action of the diffeomorphism group of M . In fact, in neighborhood of Σ gauge changes take the form

$$x'^\alpha(x^0, x^1, x^2, x^3) = x^\alpha + \frac{(x^0)^2}{2} \left(\partial_0 x'^\alpha(0, x^1, x^2, x^3) + r^\alpha(x^0, x^1, x^2, x^3) \right),$$

where r^α converges, along with its derivatives, uniformly to zero when $x^0 \rightarrow 0$. It is therefore possible to arrange the terms in parenthesis as to produce or eliminate discontinuities of $\partial_0 g'_{0\alpha}$ on Σ — showing that such terms carry no intrinsic physical meaning [L2]. Furthermore, from Assumption 2.1 one readily sees that

$$\{\{\partial_i g_{\alpha\beta}\}\} = 0 = \partial_i \{\{g_{\alpha\beta}\}\}. \quad (2.1)$$

From this and the above discussion, we see that it suffices to focus on $\partial_0 g_{ij}$ in our study of discontinuities of the metric across Σ — see definition 2.8.

We now turn our attention to the appropriate notion of weak solution for the Einstein's equations in the study of gravitational shock waves. For each fixed pair of indices $\beta\delta$, the Christoffel symbols $\Gamma_{\beta\delta}^\alpha$ locally define a vector field $\Gamma^\alpha \equiv \Gamma_{\beta\delta}^\alpha$, which in turn, under our

¹“régulièrement discontinues à la traversée de”, [L2].

²We avoid the more familiar notation $[\cdot]$ for the discontinuity of a function in that it may cause confusion when written inside an equation.

³Our convention is $(+ - - -)$.

hypotheses, is locally integrable and hence defines a vector distribution $(\Gamma_{\beta\delta}^\alpha)'$. Following Lichnerowicz, it is therefore natural to define the *curvature tensor distribution* as

$$\mathcal{R}_{\beta\gamma\delta}^\alpha = \nabla_\gamma(\Gamma_{\beta\delta}^\alpha)' - \nabla_\delta(\Gamma_{\beta\gamma}^\alpha)',$$

where the covariant derivatives are interpreted in a distributional sense [L1, L2]. To find a simple formula for the associated distributional Ricci curvature and relate it to the ordinary one, we shall use the following lemma, whose proof is an application of the tools developed in [L2].

Lemma 2.4. *Let T be a p -tensor of class C^1 on Ω^+ and Ω^- such that T and $\partial_\alpha T$ are regularly discontinuous across Σ . Then ∇T is regularly discontinuous across Σ . Furthermore, T and ∇T define tensor distributions T' and $(\nabla T)'$ such that*

$$\nabla T' - (\nabla T)' = d\varphi \otimes \delta_\Sigma \{\{T\}\},$$

where the covariant derivative $\nabla T'$ of T' is in the sense of distributions, and δ_Σ is the Dirac delta on Ω with support on Σ .

Proof. Let $A_\alpha^\beta = \Gamma_{\alpha\gamma}^\beta dx^\gamma$ be the connection one form of g on Ω . From our hypotheses, we see that A_α^β is C^1 in Ω^+ and Ω^- ; it is also regularly discontinuous across Σ . It follows that ∇T is regularly discontinuous across Σ since $\partial_\alpha T$ and T are so.

Let χ_+ (resp. χ_-) be the function defined a.e. in Ω which equals to 1 in Ω^+ (resp. Ω^-) and zero in Ω^- (resp. Ω^+). Since T and ∇T are in $L_{loc}^1(\Omega)$, they define tensor distributions, which can be written as

$$T' = \chi_+ T + \chi_- T, \tag{2.2}$$

and

$$(\nabla T)' = \chi^+ \nabla T + \chi_- \nabla T, \tag{2.3}$$

where these equalities are to be understood in the sense of distributions⁴. We can write $\text{vol}(g) = d\varphi \wedge \omega$. Notice that ω depends on φ , but if η is another 3-form such that $\text{vol}(g) = d\varphi \wedge \eta$, then $\eta = \omega + d\varphi \wedge \sigma$ for some 2-form σ . Hence, for any test function u

$$-\int_{\partial\Omega^+} u \omega = \int_{\partial\Omega^-} u \omega$$

(where $\partial\Omega^+$ and $\partial\Omega^-$ are the oriented boundaries so that $\partial\Omega^+ = -\partial\Omega^-$) has a well-defined value independent of the choice of ω . In particular, $\langle \delta_\Sigma, u \rangle$ is given by

$$\langle \delta_\Sigma, u \rangle = -\int_{\partial\Omega^+} u \omega = \int_{\partial\Omega^-} u \omega.$$

Notice that because ω is continuous this gives that δ_Σ is in fact an element of $\mathcal{D}'_{0,0}(\Omega)$. Since δ_Σ can be multiplied by locally integrable functions, this same formula also shows that δ_Σ has a well-defined action on $\mathcal{D}_{k,0}(\Omega)$, $k = 1, 2$.

⁴Equalities among quantities in $\mathcal{D}'_p(\Omega)$ are by definition in the distributional sense, so we shall no longer write “in the sense of distributions.”

Clearly $\nabla\chi_+ \in \mathcal{D}'_1(\Omega)$, and for any $u \in \mathcal{D}_1(\Omega)$

$$\begin{aligned} \langle \nabla\chi_+, u \rangle &= \langle \nabla\chi_+, u \rangle_g = -\langle \chi_+, \operatorname{div} u \rangle_g = -\int_{\Omega^+} \frac{1}{\sqrt{-|g|}} \partial_\alpha (\sqrt{-|g|} u^\alpha) \operatorname{vol}(g) \\ &= -\int_{\Omega^+} \partial_\alpha (\sqrt{-|g|} u^\alpha) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = -\int_{\partial\Omega^+} \sqrt{-|g|} u^0 dx^1 \wedge dx^2 \wedge dx^3 \\ &= -\int_{\partial\Omega^+} \varphi_\alpha u^\alpha \omega = \langle d\varphi \delta_\Sigma, u \rangle, \end{aligned}$$

so that $\nabla\chi_+ = d\varphi \delta_\Sigma$. Similarly, one finds $\nabla\chi_- = -d\varphi \delta_\Sigma$. Therefore,

$$\nabla(\chi^+ T) = (\nabla\chi_+) \otimes T + \chi_+ \nabla T = d\varphi \delta_\Sigma \otimes T + \chi_+ \nabla T.$$

Computing a similar expression for $\nabla(\chi^+ T)$ and using (2.2) and (2.3) yields the result. \square

It is seen that the discontinuities of g enter in the expression for the distributional curvature. The following lemma is useful to handle them.

Lemma 2.5. *For each of the functions $g_{\alpha\beta}$, there exists a scalar distribution $d_{\alpha\beta}$ such that*

$$\delta_\Sigma \{ \{ \partial_\gamma g_{\alpha\beta} \} \} = \varphi_\gamma d_{\alpha\beta}.$$

Proof. Arguing similarly to the proof of Lemma 2.4 we see that

$$\begin{aligned} \langle \nabla_i \delta_\Sigma, u \rangle &= \int_{\partial\Omega^+} \partial_i (\sqrt{-|g|} u) dx^1 \wedge dx^2 \wedge dx^3 \\ &= \int_{\Omega^+} \partial_i \partial_0 (\sqrt{-|g|} u) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= 0, \end{aligned}$$

where we used that u , being a test function, is compactly supported in Ω . A similar result holds in Ω^- and we conclude that $\nabla_i \delta_\Sigma = 0$ (in a more invariant fashion, we can say $\nabla_X \delta_\Sigma = 0$ for vector fields X tangent to $\varphi = \text{constant}$). It follows that

$$\nabla \delta_\Sigma = d\varphi \nabla_0 \delta_\Sigma.$$

This combined with (2.1) (and, of course, our assumptions on g) produces the result. \square

Intuitively, $d_{\alpha\beta}$ corresponds to the “jumps” that come when we try to differentiate $\delta_\Sigma \{ \{ g_{\alpha\beta} \} \}$ across Σ . It is not difficult to see that the existence of $d_{\alpha\beta}$ is equivalent to the formulation given in [L2], namely, to find, for each $g_{\alpha\beta}$, a function $b_{\alpha\beta}$ defined over Σ such that

$$\{ \{ \partial_\gamma g_{\alpha\beta} \} \} = \varphi_\gamma b_{\alpha\beta} \text{ on } \Sigma$$

and satisfying⁵ $d_{\alpha\beta} = \delta_\Sigma b_{\alpha\beta}$.

Remark 2.6. $b_{\alpha\beta}$ depends on the choice of equation for Σ . The functions $b_{\alpha\beta}$ were initially introduced by Lichnerowicz [L2] and are important in determining the conditions of shock for gravitational waves. Such conditions will appear below when we construct the propagator for the Klein-Gordon equation.

⁵Notice that, in principle, $b_{\alpha\beta}$ is defined only on Σ , but this suffices since δ_Σ is supported on Σ ; alternatively $b_{\alpha\beta}$ can be extended to Ω without affecting the distributional equality $d_{\alpha\beta} = \delta_\Sigma b_{\alpha\beta}$.

From Lemma 2.4, Lemma 2.5 and the above considerations it follows that

$$\mathcal{R}_{\alpha\beta\gamma\delta} = R'_{\alpha\beta\gamma\delta} + \delta_\Sigma \mathcal{H}_{\alpha\beta\gamma\delta},$$

where the distribution $R'_{\alpha\beta\gamma\delta}$, with $R_{\alpha\beta\gamma\delta}$ being the curvature tensor, is well-defined because of Assumption 2.1, and $\mathcal{H}_{\alpha\beta\gamma\delta}$ is the tensor on Σ given by

$$-2\mathcal{H}_{\alpha\beta\gamma\delta} = b_{\alpha\gamma}\varphi_\beta\varphi_\delta - b_{\alpha\delta}\varphi_\beta\varphi_\gamma + b_{\beta\delta}\varphi_\alpha\varphi_\gamma - b_{\beta\gamma}\varphi_\alpha\varphi_\delta.$$

The distributional Ricci tensor is defined as

$$\mathcal{R}_{\alpha\beta} = R'_{\alpha\beta} + \delta_\Sigma \mathcal{H}_{\alpha\beta}, \quad (2.4)$$

where

$$2\mathcal{H}_{\alpha\beta} = b_{\alpha\mu}\varphi^\mu\varphi_\beta + b_{\beta\mu}\varphi^\mu\varphi_\alpha - b_\mu^\mu\varphi_\alpha\varphi_\beta - b_{\alpha\beta}\varphi^\mu\varphi_\mu, \quad (2.5)$$

so that $\mathcal{R}_{\alpha\beta}$ is formally the trace of the distributional curvature tensor.

Recall that given a stress-energy tensor $\mathcal{T}_{\alpha\beta}$ (possibly identically zero), one can write Einstein's equations as

$$R_{\alpha\beta} = \kappa\rho_{\alpha\beta}, \quad (2.6)$$

where κ is a constant and

$$\rho_{\alpha\beta} = \mathcal{T}_{\alpha\beta} - \frac{1}{2}\mathcal{T}g_{\alpha\beta},$$

with \mathcal{T} the trace of $\mathcal{T}_{\alpha\beta}$. The last ingredient we need to define the distributional Einstein's equations is the regularity of $\mathcal{T}_{\alpha\beta}$.

Assumption 2.2. $\mathcal{T}_{\alpha\beta}$ is a given symmetric two-tensor, continuous on Ω^+ and Ω^- and regularly discontinuous across Σ .

It follows that $\rho_{\alpha\beta}$ shares the same regularity properties of $\mathcal{T}_{\alpha\beta}$. The reason why we think of $\mathcal{T}_{\alpha\beta}$ as given is that existence of solutions to the distributional Einstein's equations — in which case $\mathcal{T}_{\alpha\beta}$ has a determined functional form but depends on the metric and the matter fields of the problem — will not be investigated; rather we shall assume we are given a solution that defines a background on which fields will be quantized.

Definition 2.7. The distributional Einstein's equations are defined as

$$\mathcal{R}_{\alpha\beta} = \kappa\rho'_{\alpha\beta}. \quad (2.7)$$

Definition 2.8. The hypersurface Σ is called a *wave front* and is said to define a *gravitational shock wave* if g satisfies (2.7), and the first derivatives of the physical components of g are discontinuous across Σ , and also regularly discontinuous across Σ .

It is possible to show that a shock wave Σ is necessarily light-like⁶; had it not been light-like, and hence characteristic to the reduced Einstein equations, the values of the induced metric and its derivatives would uniquely determine a regular solution on both sides of Σ , preventing discontinuities. See [L2] for details.

⁶Recall that hydrodynamic shock waves have the property of being supersonic before the shock and subsonic after the shock, with ordinary waves propagating at the sound speed. Here, the speed of light plays the role of the sound speed, hence shocks cannot be “superluminal” before the shock.

3. QUANTIZATION.

In this section, we adapt the techniques of [L3] to the weak setting developed above. We assume the same hypotheses and notation as before. We shall deal only with a free scalar field, although it is very likely such results can be generalized to tensor and spinor fields. Not surprisingly, the extension to interacting theories, on the other hand, is expected to pose severe difficulties. For the rest of the paper, we let $m > 0$ be a fixed parameter. For a compact set $K \subset \Omega$, we denote by $C^+(K)$ (resp. $C^-(K)$) the future (resp. past) of K in the usual sense of GR, and the cone of K the set $C(K) = C^+(K) \cup C^-(K)$. As our point of view is purely local, strictly speaking $C^+(K)$, is the future of K within Ω (analogously for $C^-(K)$).

Proposition 3.1. *For each fixed $x \in \Omega$, there exist two elementary kernels E_x^\pm satisfying*

$$(\square_g + m^2)E_x^\pm = \delta_x,$$

where δ_x is the Dirac delta supported at x . E_x^+ (resp. E_x^-) is unique and has support on $C^+(x)$ (resp. $C^-(x)$).

Remark 3.2. In Minkovski space, E^\pm are the standard advanced and retarded kernels.

Proof. We give the proof for E^+ , with the existence of E^- being completely analogous. First, notice that the standard formula

$$\int_{\Omega} u \square_g v \, \text{vol}(g) = \int_{\Omega} u \partial_{\mu} (g^{\mu\nu} \sqrt{-|g|} \partial_{\nu} v) \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \int_{\Omega} v \square_g u \, \text{vol}(g)$$

still holds for all $u, v \in \mathcal{D}_{2,0}(\Omega)$, despite g being only continuous on the whole of Ω . \square_g is, therefore, formally self-adjoint, hence we seek to find E_x^+ such that for all $u \in \mathcal{D}_{2,0}(\Omega)$

$$\langle E_x^+, (\square_g + m^2)u \rangle = \langle \delta_x, u \rangle.$$

Following Choquet-Bruhat [B1], we construct a parametrix σ_x^+ , i.e., a distribution satisfying⁷

$$(\square_g + m^2)\sigma_x^+ = \delta_x - Q_x^+, \tag{3.1}$$

where Q_x^+ is an integrable function supported on $\Gamma_x \equiv \partial C^+(x)$. Indeed, we shall show that the fixed point argument employed in [B1], where it is assumed that the metric is sufficiently differentiable, still goes through under our assumptions.

We start by investigating the restrictions that (2.7) imposes on the quantities $b_{\alpha\beta}$ — these are the so-called conditions of shock, originally studied in [L2]. Because g is C^2 over Ω^+ and Ω^- , and in light of Assumption 2.1, we have that (2.6) is satisfied in $\Omega^+ \cup \Omega^-$ and, moreover,

$$R'_{\alpha\beta} = \kappa \rho'_{\alpha\beta} \text{ in } \Omega.$$

From this, (2.4), (2.5) and (2.7) we conclude that

$$\varphi^\mu b_{[\alpha|\mu|\varphi\beta]} = 0 \text{ on } \Sigma. \tag{3.2}$$

⁷The terminology “parametrix” was introduced by Hilbert and usually indicates an approximation for the fundamental solution of a linear equation.

We now claim that $\varphi^\mu \nabla_\mu$ is a well-defined operator along Σ . In fact, direct computation gives that on Σ

$$\varphi^\mu \{ \{ \Gamma_{\alpha\mu}^\beta \} \} = \varphi^\mu b_{[\alpha|\mu|} \varphi_{\beta]},$$

which vanishes by (3.2). From this, the regularity of φ in Assumption 2.1 and the fact that Σ is light-like [L2], it follows not only that $\varphi^\mu \nabla_\mu$ is well-defined over Σ but also that

$$\varphi^\mu \nabla_\mu \varphi^\alpha = 0 \text{ on } \Sigma.$$

As a consequence, geodesics are well-defined over (and in fact span) Σ . Hence, even under the circumstance $\Gamma_x \cap \Sigma \neq \emptyset$, we can set up a system of ordinary differential equations over Γ_x that determines σ_x^+ ; this is done exactly as in [B1], which the reader is referred to for details.

From equality (3.1), it now follows that

$$u(x) = \int_{\Omega} Q_x^+ u \, \text{vol}(g) + \langle \sigma_x^+, (\square_g + m^2)u \rangle.$$

This can be viewed as an integral equation for u , which can be solved by a Neumann-series type of technique as in chapter V of [dR], possibly after shrinking Ω . The solution is unique and operates linearly and continuously over u , yielding the desired kernel. \square

As usual, we think of E_x^\pm as defining a distribution $E^\pm(x', x)$ in $\Omega \times \Omega$ by

$$\langle E^\pm(x', x), u(x', x) \rangle_{\Omega \times \Omega} = \langle E_x^\pm(x), u(x', x) \rangle_{\Omega \times \Omega}. \quad (3.3)$$

The lemma below verifies that the standard symmetry properties of these distributions known to be true in the smooth setting, continue to hold under our hypothesis.

Lemma 3.3. *Let $E^\pm(x, x')$ be as above. Then*

$$E^+(x, x') = E^-(x', x),$$

for $x, x' \in \Omega$.

Proof. Fix a test function u and define

$$v_1(x') = \langle E^-(x, x'), u(x) \rangle.$$

Since $E_x^-(x')$ has support on the past of x , v_1 is supported on the past of the support of u . We have

$$(\square_{x'} + m^2)v_1(x') = \langle (\square_{x'} + m^2)E^-(x, x'), u(x) \rangle = \langle \delta(x, x'), u(x) \rangle = u(x'),$$

where $\square \equiv \square_g$, \square_x means that derivatives are with respect to the x variable, and we used the symmetry of the Dirac delta.

For sufficiently smooth metrics, Choquet-Bruhat proved [B2] that any solution v_2 of

$$(\square + m^2)v_2 = u \quad (3.4)$$

with support compact towards the future⁸ is given by

$$v_2(x') = \langle E^+(x', x), u(x) \rangle, \quad (3.5)$$

⁸ $K \subset \Omega$ is said compact towards the future if $C^-(K) \cap C^+(x)$ is compact or empty for each $x \in \Omega$; a similar notion applies for compact towards the past.

provided that Ω is taken sufficiently small. An inspection on her proof shows that the same statement still holds in our setting. In fact, the set $K = C_-(\text{supp}(u)) \cap C_+(x)$ is compact (or empty). Choose a function $z \in \mathcal{D}_{2,0}(\Omega)$ equal to 1 in a compact neighborhood of K . If v_2 satisfies (3.4), we obtain that for any $x' \in C_-(\text{supp}(u))$,

$$\langle E^+(x', x), u(x) \rangle = \langle E^+(x', x), (\square_x + m^2)v_2(x) \rangle = \langle E^+(x', x), (\square_x + m^2)(z(x)v_2(x)) \rangle.$$

But, on the other hand, if

$$\tilde{v}_2(x') = \langle E^+(x', x), u(x) \rangle,$$

then

$$\langle E^+(x', x), u(x) \rangle = \langle (\square_x + m^2)E^+(x', x), z(x)v_2(x) \rangle = \tilde{v}_2(x'),$$

which shows (3.5). We conclude that $v_1 = v_2$, from what the result follows. \square

Following standard convention and terminology, we then define the *propagator* of $\square_g + m^2$ as the distribution on $\Omega \times \Omega$ given by

$$G(x, x') = E^+(x', x) - E^-(x', x).$$

It is seen that

$$(\square_x + m^2)G(x, x') = 0.$$

In light of Lemma 3.3,

$$G(x, x') = -G(x', x).$$

Consider the massive wave equation in Ω ,

$$(\square_g + m^2)u = 0. \tag{3.6}$$

Let us denote by $D^+(K)$ (resp. $D^-(K)$) the future (resp. past) domain of dependence of an achronal set K , and $D(K) = D^+(K) \cup D^-(K)$. We next suppose that:

Assumption 3.1. There exists in Ω a kernel $\mathcal{G}(x, x')$, which is a solution (in x) of (3.6), satisfying $\mathcal{G}(x, x') = \mathcal{G}(x', x)$, and such that, for each space-like three surface $\mathcal{S} \subset \Omega$ and $x, x' \in D(\mathcal{S}) \cap \Omega$, the following holds:

$$G(x, x') = \int_{\mathcal{S}} \left(\mathcal{G}(x, y) \partial_\mu \mathcal{G}(x', y) - \mathcal{G}(x', y) \partial_\mu \mathcal{G}(x, y) \right) dA_g^\mu(y), \tag{3.7}$$

where dA_g is the volume element induced on \mathcal{S} by g .

Remark 3.4. In Minkovski space, \mathcal{G} is the so-called D^1 distribution associated with the Pauli-Jordan propagator [BS].

As in the usual case of smooth coefficients, for $x \in D(\mathcal{S}) \cap \Omega$ we have the following formula for a solution u of the Cauchy problem of (3.6) [B1, B2],

$$u(x) = \int_{\mathcal{S}} \left(u(y) \partial_\mu G(x, y) - G(x, y) \partial_\mu u(y) \right) dA_g^\mu(y).$$

This determines the solution u in terms of the Cauchy data $q := u|_{\Sigma}$ and $p := \partial_{\nu} u|_{\Sigma}$. The pair (q, p) plays the role of a point on phase space. We now define \tilde{u} , depending on u , by

$$\tilde{u}(x) = \int_S \left(u(y) \partial_{\mu} \mathcal{G}(x, y) - \mathcal{G}(x, y) \partial_{\mu} u(y) \right) dA_g^{\mu}(y),$$

and introduce the norm

$$(u, u) = \int_S \left(u(y) \partial_{\mu} \tilde{u}(y) - \tilde{u}(y) \partial_{\mu} u(y) \right) dA_g^{\mu}(y).$$

Define

$$G^+ = \frac{G - i\mathcal{G}}{2}, \quad G^- = \frac{G + i\mathcal{G}}{2},$$

and

$$u^+ = \frac{u - i\tilde{u}}{2}, \quad u^- = \frac{u + i\tilde{u}}{2}.$$

Proposition 3.5. *The following identity holds*

$$(u^+, u^+) = (u^-, u^-) = \frac{1}{2}(u, u).$$

Proof. This follows directly from the above formulas. □

We can now carry out the quantization of a free scalar field over the background (Ω, g) by following the corresponding steps in the quantization of fields defined over a smooth curved background, as originally proposed by Lichnerowicz [L3], and extended by other authors, in particular Wald [W] (see also [HW]). In order to obtain a well-defined quantization procedure, one has to be specific about the structure of the operators involved, indicating their domains, self-adjointness properties etc. However, with the propagator $G(x, x')$, Proposition 3.5, and formula (3.7) at hand, these constructions are the same as in [L3] (see also [W]). Hence, for the sake of simplicity, we shall restrict ourselves to indicating what the main features of the corresponding quantum theory are.

We have:

I. The association $u \mapsto \tilde{u}$ defines a linear automorphism J on the space of solutions⁹ of (3.6). u^{\pm} are eigenfunctions of J corresponding to the eigenvalues $\pm i$. The projections $u \mapsto u^{\pm}$ correspond exactly to the decomposition of the field into positive and negative energies.

⁹A 2-form ω can also be introduced in the space of solutions, and shown to define a symplectic structure. ω and J are compatible and define an (infinite dimensional) Kähler structure. These intrinsic quantities can be used to derive an appropriate notion of inner product to carry out the quantization procedure. See [L3] and [W] for details.

II. Assume from now on that u is operator-valued. Our definitions and formula (3.7) imply

$$\begin{aligned} [u^+, u^+] &= 0 = [u^-, u^-], \\ [u^+(x), u^-(x')] &= -iG^+(x, x') \text{id}, \\ [u^-(x), u^+(x')] &= -iG^-(x, x') \text{id}, \\ [u(x), u(x')] &= -iG(x, x') \text{id}, \end{aligned}$$

where id is the identity operator, and the pointwise commutators are interpreted as formally expressing the corresponding distributional identity, e.g.

$$[u(f), u(h)] = -i \int_{\Omega \times \Omega} f(x) G(x, x') h(x') \sqrt{-|g|(x)} \sqrt{-|g|(x')} dx dx',$$

where f and g are test functions.

III. A theory of creation and annihilation operators can be constructed from these formulas and (3.7). All the usual propagators can be deduced from the elementary kernels and \mathcal{G} .

IV. In Minkovski space, the above decomposition agrees with the usual split of u into negative and positive energy solutions via Fourier transform.

One feature that stands out is the existence of positive energy solutions. This seems to be in contradiction with the established fact that in a general spacetime there is no natural notion of positive frequency (which is itself a consequence of the lack of uniqueness for the vacuum state in such situations). Recall, however, that such a notion is well defined for spacetimes that are (i) globally hyperbolic and (ii) stationary. The former property has been implicitly employed by addressing the problem solely from a local perspective and considering $x, x' \in D(\mathcal{S}) \cap \Omega$ in Assumption 3.1. These are not essential: when considering a global point of view, it is rather natural to restrict oneself to globally hyperbolic spacetimes, and our lemmas and propositions can be extended to this setting (recall that we focused on local constructions because the approximation arguments we used are local in nature).

Although not implicitly assumed, property (ii) was “almost” assumed, in the following sense. We have supposed the existence of the kernel \mathcal{G} , but sufficient conditions that imply the validity of Assumption 3.1 have not been given. It can be shown, however, that \mathcal{G} in fact exists for globally hyperbolic stationary spacetimes (Euclidean at infinity for non-compact Cauchy surfaces) [M] (see also [C]). But, in the special case of globally hyperbolic stationary spacetimes, the decomposition into positive and negative energy solutions does in fact hold true [W]. Since it is not known, however, whether global hyperbolicity and stationarity are also necessary conditions for the existence of \mathcal{G} [L3], here we preferred to take the slightly more general point of view of postulating the existence of \mathcal{G} itself.

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